MA 335, Spring 2014

Assignment 8.

This homework is due *Friday* March 28.

There are total 24 points in this assignment. 21 points is considered 100%. If you go over 21 points, you will get over 100% for this homework (but not over 115%) and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge.* Your solutions should contain full proofs. Bare answers will not earn you much.

1. Möbius function

(1) [3pt] (6.2.2) The *Mangoldt function* Λ is defined by

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^k, \text{ where } p \text{ is prime and } k \geq 1; \\ 0, & \text{othewise.} \end{cases}$$

Prove that $\Lambda(n) = \sum_{d|n} \mu(n/d) \log d = -\sum_{d|n} \mu(d) \log d$.

(*Hint:* Show $\sum_{d|n} \Lambda(d) = \log n$ and then apply the Möbius inversion formula.)

(2) [3pt] (6.2.3) Let $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ be the prime factorization of the integer $n > 1$. If f is a multiplicative function that is not identically zero, prove that

$$\sum_{d|n} \mu(d) f(d) = (1 - f(p_1))(1 - f(p_2)) \cdots (1 - f(p_r)).$$

(*Hint:* Show that left hand side is multiplicative. Check that the equality holds for $n = p^k$ and argue that it suffices.)

(3) (Part of 6.2.4) For the integer $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, use Problem 2 to establish the following:

(a) [1pt] $\sum_{d|n} \mu(d)\tau(d) = (-1)^r$,

(b) [1pt] $\sum_{d|n} \mu(d)\sigma(d) = (-1)^r p_1 p_2 \cdots p_r$,

2. Euler's Function, Euler's Theorem

(4) [2pt] (7.2.2) Verify that $\varphi(n) = \varphi(n+1) = \varphi(n+2)$ for $n = 5186$.

(5) [3pt] (7.2.10) If every prime that divides n also divides m, establish $\varphi(mn) = n\varphi(m)$; in particular, $\varphi(n^2) = n\varphi(n)$ for every positive integer n.

(6) [2pt] (7.2.13) Prove that if $d \mid n$, then $\varphi(d) \mid \varphi(n)$. (*Hint:* Work with the prime factorizations of d and n.)

(7) [2pt] (7.3.9) Use Euler's theorem to evaluate remainder of 2^{100000} mod 77.

(8) Solve the congruence $x^{11} \equiv 26 \pmod{63}$ using the following procedure.

(a) [1pt] Argue that if x is a solution, then $\gcd(x, 63) = 1$

(b) [3pt] Raise both sides to the power 23 using Euler's theorem to find LHS and using square-and-multiply to find RHS.

(9) [3pt] Solve the congruence $x^{13} \equiv 17 \pmod{75}$ similarly to Problem 8. (*Hint:* Start by finding n such that $13n \equiv 1 \pmod{\varphi(75)}$. Then raise the congruence to that power.)